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Critical Evaluation of Two-Equation Models for Near-Wall Turbulence

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A variety of two-equation turbulence models—including several versions of the K-ϵ model as well as the K-ω model—are analyzed critically for near wall turbulent flows from a theoretical and computational standpoint. It is shown that the K-ϵ model has two major problems associated with it: the lack of natural boundary conditions for the dissipation rate and the appearance of higher-order correlations in the balance of terms for the dissipation rate at the wall. Insofar as the former problem is concerned, either physically inconsistent boundary conditions have been used or the boundary conditions for the dissipation rate have been tied to higher-order derivatives of the turbulent kinetic energy, which leads to numerical stiffness. The K-ω model can alleviate these problems since the asymptotic behavior of ω is known in more detail and since its near-wall balance involves only exact viscous terms. However, the modeled form of the ω equation used in the literature is incomplete: an exact viscous term is neglected, which causes the model to behave in an asymptotically inconsistent manner. By including this viscous term, and by introducing new wall damping functions with improved asymptotic behavior, a new K-τ model (where $\tau \equiv 1/\omega$ is the turbulent time scale) is developed. It is demonstrated that this new model yields improved predictions for turbulent boundary layers.

Introduction

An increasing number of practical engineering calculations of turbulent flows have been based on two-equation turbulence models. For many technologically important turbulent flows, two-equation models represent a nice compromise between zero- or one-equation models and second-order closures (the former models tend to require too many ad hoc empiricisms, whereas second-order closure models can be overly expensive for design calculations). The K-ϵ model[1] is the most popular two-equation turbulence model in use today. When utilized in conjunction with wall functions, the K-ϵ model is reasonably well behaved and has been applied to the solution of a variety of engineering problems with a moderate amount of success. However, many important technological applications require the integration of turbulence models directly to a solid boundary, particularly in problems where wall transport properties are needed or where there is flow separation. The problem of developing low-Reynolds-number near-wall corrections to the K-ϵ model that can be robustly and accurately integrated to a solid boundary remains unresolved so that models along alternative lines continue to be proposed (see Ref. 2 for a recent review). Most of these near-wall K-ϵ models involve an excessive amount of ad hoc empiricisms and are numerically stiff in turbulent boundary-layer flows. This motivated some researchers to pursue alternative two-equation models based on a modeled transport equation for the turbulent time scale. The most notable example is the K-ω model of Wilcox and Traci[3] and Wilcox,[4] where modeled transport equations for the turbulent kinetic energy K and reciprocal turbulent time scale ω are solved. There is considerable evidence that the K-ω model is more computationally robust than the K-ϵ model for the integration of turbulent flows to a solid boundary. However, the K-ω model yields solutions for the turbulent kinetic energy that are asymptotically inconsistent near a solid boundary.[4] Hence, there is the need to re-examine this problem from a basic theoretical and computational standpoint. This establishes the motivation for the present paper.

In this paper, the near-wall asymptotics of two-equation turbulence models will be examined from a basic theoretical standpoint. It will be shown that the K-ϵ model has two major problems associated with it. The first arises from the lack of natural boundary conditions for the turbulent dissipation rate, which has caused modelers to use a variety of derived boundary conditions that are either asymptotically inconsistent (e.g., the boundary condition of vanishing normal derivative of dissipation) or numerically stiff (e.g., the boundary condition that ties the dissipation to higher-order derivatives of the turbulent kinetic energy). The second problem, which can be the source of substantial inaccuracies and numerical stiffness, is tied to the fact that the balance of terms at the wall in the modeled dissipation rate transport equation depends on higher-order correlations whose models have considerable uncertainties.

It will be demonstrated that both of these problems can be largely alleviated by solving a modeled transport equation for the turbulent time scale $\tau \equiv K/\epsilon$ since 1) near the wall, $\tau \doteq y^2/2\nu$, which provides the needed natural boundary conditions,

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and, 2) the balance of terms at the wall in the modeled transport equation for τ involves only *exact* viscous terms. It will be argued that these features are primarily responsible for the more computationally robust performance of the K-ω model of Refs. 3 and 4. However, the K-ω model yields results for the turbulent kinetic energy—as well as other turbulence quantities—that are asymptotically inconsistent (e.g., near the wall, $K \sim y^{3.23}$ instead of the expected $K \sim y^2$ behavior). It will be proven that this problem arises because an exact viscous cross-diffusion term has been neglected in the modeled ω-transport equation without compensating for its near-wall effect—a decision made for computational simplicity. A new K-τ model is obtained by including this exact viscous term and by substituting improved wall damping functions obtained by an asymptotic analysis using the results of direct numerical simulations of turbulent channel flow.[5] The new model will be tested for the flat-plate turbulent boundary layer, and comparisons will be made with the predictions of other models (i.e., the K-ϵ models of Refs. 6 and 7) as well as the K-ω model[4] to assess its performance.

Near-Wall Asymptotic Analysis

For simplicity, we will restrict our analysis to incompressible turbulent flows (however, the crucial conclusions that will be drawn carry over to compressible flows). The mean velocity $\bar{u}$ and the mean pressure $\bar{p}$ are solutions of the Reynolds-averaged Navier-Stokes and continuity equations given by

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = -\frac{\partial \bar{p}}{\partial x_i} + \nu \nabla^2 \bar{u}_i + \frac{\partial \tau_{ij}}{\partial x_j} \tag{1}$$

$$\frac{\partial \bar{u}_i}{\partial x_i} = 0 \tag{2}$$

where $\tau_{ij} = -\overline{u_i' u_j'}$ is the Reynolds stress tensor, ν is the kinematic viscosity, and the usual Einstein summation convention applies to repeated indices. We will consider the commonly used two-equation models based on an eddy viscosity, where

$$\tau_{ij} = -\tfrac{2}{3} K \delta_{ij} + \nu_T \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) \tag{3}$$

$$\nu_T = C_\mu (K^2/\epsilon) \tag{4}$$

given that $K \equiv \tfrac{1}{2}\overline{u_i' u_i'}$ is the turbulent kinetic energy, $\epsilon = \nu \overline{\partial u_i'/\partial x_j \partial u_i'/\partial x_j}$ is the turbulent dissipation rate, and C_μ is a dimensionless constant at high-turbulence Reynolds numbers. In two-equation models, transport equations are solved for any two linearly independent variables constructed from K and ϵ. In the K-ϵ model, modeled transport equations for K and ϵ are solved; in the K-ω model, modeled transport equations for K and the reciprocal turbulent time scale $\omega \equiv \epsilon/K$ are solved; and in the K-τ model, modeled transport equations for K and the turbulent time scale $\tau \equiv K/\epsilon$ are solved. The exact transport equations for K and ϵ are as follows[8]:

$$\frac{DK}{Dt} = \tau_{ij} \frac{\partial \bar{u}_i}{\partial x_j} - \epsilon - \mathfrak{D} + \nu \nabla^2 K \tag{5}$$

$$\frac{D\epsilon}{Dt} = \mathcal{P}_\epsilon - \Phi_\epsilon - \mathfrak{D}_\epsilon + \nu \nabla^2 \epsilon \tag{6}$$

where $D/Dt = \partial/\partial t + \bar{u} \cdot \nabla$. In Eqs. (5) and (6),

$$\mathfrak{D} = \frac{\partial}{\partial x_i} \left(\frac{1}{2} \overline{u_j' u_j' u_i'} + \overline{p' u_i'} \right) \tag{7}$$

$$\mathfrak{D}_\epsilon = 2\nu \frac{\partial}{\partial x_i} \left(\overline{\frac{\partial p'}{\partial x_j} \frac{\partial u_i'}{\partial x_j}} \right) + \nu \frac{\partial}{\partial x_j} \left(\overline{u_j' \frac{\partial u_i'}{\partial x_k} \frac{\partial u_i'}{\partial x_k}} \right) \tag{8}$$

are turbulent transport terms, and

$$\begin{aligned} \mathcal{P}_\epsilon = & -2\nu \overline{\frac{\partial u_i'}{\partial x_j} \frac{\partial u_k'}{\partial x_j}} \frac{\partial \bar{u}_i}{\partial x_k} - 2\nu \overline{\frac{\partial u_j'}{\partial x_i} \frac{\partial u_j'}{\partial x_k}} \frac{\partial \bar{u}_i}{\partial x_k} \\ & - 2\nu \overline{\frac{\partial u_i'}{\partial x_k} \frac{\partial u_i'}{\partial x_j} \frac{\partial u_k'}{\partial x_j}} + 2\nu \overline{u_k' \frac{\partial u_i'}{\partial x_j}} \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_k} \end{aligned} \tag{9}$$

$$\Phi_\epsilon = 2\nu^2 \overline{\frac{\partial^2 u_i'}{\partial x_j \partial x_k} \frac{\partial^2 u_i'}{\partial x_j \partial x_k}} \tag{10}$$

are, respectively, the production and destruction of dissipation terms.

The Taylor series expansions for the components of $u_i' \equiv (u', v', w')$ are as follows near a wall:

$$u' = a_1 y + a_2 y^2 + \cdots \tag{11}$$

$$v' = b_2 y^2 + b_3 y^3 + \cdots \tag{12}$$

$$w' = c_1 y + c_2 y^2 + \cdots \tag{13}$$

where $a_i = a_i(x,z,t)$, $b_i = b_i(x,z,t)$, and $c_i = c_i(x,z,t)$ given that the coordinate y is normal to the wall (later, wall coordinates will be used where $y^+ = yu_\tau/\nu$ given that u_τ is the shear velocity). Of course, the no-slip condition at the wall eliminates a_0, b_0, and c_0, whereas the continuity equation eliminates b_1 (cf. Ref. 8). It is a straightforward matter to show that near a wall,

$$K = \mathcal{O}(y^2), \qquad \epsilon = \mathcal{O}(1), \qquad \tau = \mathcal{O}(y^2) \tag{14}$$

$$\frac{\partial \bar{u}}{\partial y} = \mathcal{O}(1), \qquad \overline{u'^2} = \mathcal{O}(y^2), \qquad \overline{u'v'} = \mathcal{O}(y^3) \tag{15}$$

$$\overline{v'^2} = \mathcal{O}(y^4), \qquad \overline{w'^2} = \mathcal{O}(y^2), \qquad \mathcal{P} = \mathcal{O}(y^3) \tag{16}$$

$$\mathfrak{D} = \mathcal{O}(y), \qquad \mathcal{P}_\epsilon = \mathcal{O}(y), \qquad \Phi_\epsilon = \mathcal{O}(1) \tag{17}$$

$$\mathfrak{D}_\epsilon = \mathcal{O}(1), \qquad \nabla^2 K = \mathcal{O}(1), \qquad \nabla^2 \epsilon = \mathcal{O}(1) \tag{18}$$

where $\mathcal{P} \equiv \tau_{ij} \partial \bar{u}_i / \partial x_j$ is the turbulence production.

An asymptotic analysis of the K-ϵ model will be conducted first. In the K-ϵ model, the eddy viscosity near a wall is taken to be of the form

$$\nu_T = C_\mu f_\mu (K^2/\epsilon) \tag{19}$$

The asymptotic analysis presented in this section indicates that $f_\mu = \mathcal{O}(1/y)$ near the wall since, due to Eq. (15), ν_T must be of $\mathcal{O}(y^3)$ in this region. Of course, sufficiently far from the wall, f_μ assumes a value of 1. (C_μ is a constant typically taken to be 0.09.) The turbulent transport term $\mathfrak{D}$ in the kinetic energy equation (5) is modeled using a gradient transport hypothesis:

$$\mathfrak{D} = -\frac{\partial}{\partial x_i} \left(\frac{\nu_T}{\sigma_K} \frac{\partial K}{\partial x_i} \right) \tag{20}$$

where σ_K is a constant. From Eqs. (14), (17), and (19), it is clear that this model is *not* asymptotically consistent. However, $\mathfrak{D}$ consists of two parts—the triple-velocity term and the pressure diffusion term—as given by Eq. (7). Direct numerical simulations of the Navier-Stokes equations indicate that

$$\frac{\partial}{\partial x_i} (\overline{pu_i'}) \ll \frac{\partial}{\partial x_i} \left(\frac{1}{2} \overline{u_i' u_j' u_j'} \right)$$

except very close to the wall (i.e., inside of $y^+ = 2$; see Ref. 5), and in this region $\mathfrak{D}$ is negligible in comparison to the dissipation rate and the viscous diffusion of the turbulent kinetic energy. Hence, if we approximate $\mathfrak{D}$ by

$$\mathfrak{D} \doteq \frac{\partial}{\partial x_i}\left(\frac{1}{2}\overline{u_j' u_j' u_i'}\right) \tag{21}$$

then the gradient transport model (20) is asymptotically consistent since the right-hand sides of Eqs. (20) and (21) are both of $\mathcal{O}(y^3)$ as the wall is approached. Hence, it would appear that the asymptotic errors introduced by the use of Eq. (20) in the K-equation model are probably not that serious.

The turbulent transport term $\mathfrak{D}_\epsilon$ in the dissipation rate transport equation is also modeled by a gradient transport hypothesis:

$$\mathfrak{D}_\epsilon = -\frac{\partial}{\partial x_i}\left(\frac{\nu_T}{\sigma_\epsilon}\frac{\partial \epsilon}{\partial x_i}\right) \tag{22}$$

(where σ_ϵ is a constant) in the K-ϵ model. This model is not asymptotically consistent since $\mathfrak{D}_\epsilon = \mathcal{O}(1)$ near a wall, while the right-hand side of Eq. (22) is $\mathcal{O}(y^2)$. However, this inconsistency is probably not of great consequence since both $\mathfrak{D}_\epsilon$ and Φ_ϵ are of $\mathcal{O}(1)$ near a wall, but direct numerical simulations of turbulent channel flow indicate that $\mathfrak{D}_\epsilon \ll \Phi_\epsilon$ (cf. Ref. 5).

The production of dissipation $\mathcal{P}_\epsilon$ and the destruction of dissipation Φ_ϵ are modeled as follows:

$$\mathcal{P}_\epsilon = C_{\epsilon 1} f_1 \frac{\epsilon}{K}\tau_{ij}\frac{\partial \bar{u}_i}{\partial x_j} \tag{23}$$

$$\Phi_\epsilon = C_{\epsilon 2} f_2 \frac{\epsilon^2}{K} \tag{24}$$

in the K-ϵ model where the wall damping functions $f_1, f_2 \to 1$ away from the wall. It is clear from Eqs. (14–17) that these models are asymptotically consistent if $f_2 = \mathcal{O}(1)$ and $f_2 = \mathcal{O}(y^2)$ near a wall. It thus follows that the K-ϵ model will generate solutions for K, ϵ and $\overline{u'v'}$ that are asymptotically consistent if the damping functions $f_\mu = \mathcal{O}(1/y)$ and $f_2 = \mathcal{O}(y^2)$ near a wall with $f_1 = 1$.

Although the K-ϵ model can be made asymptotically consistent in near-wall turbulent flows by the introduction of only two wall damping functions—namely, $f_\mu = \mathcal{O}(1/y)$ and $f_2 = \mathcal{O}(y^2)$—there are still some other major problems that need to be discussed. There are no natural boundary conditions on ϵ; consequently, boundary conditions must be either derived or postulated. One of the commonly used derived boundary conditions is

$$\nu\frac{\partial^2 K}{\partial y^2} = \epsilon \tag{25}$$

at the wall, which is a rigorous consequence of the exact transport equation for K. Equation (25) requires information at the wall on the second-order derivative of the turbulent kinetic energy—a feature that can lead to considerable numerical stiffness. Some of the stiffness can be alleviated by utilizing the alternative version of Eq. (25):

$$2\nu\left(\frac{\partial\sqrt{K}}{\partial y}\right)^2 = \epsilon \tag{26}$$

at the wall. However, even Eq. (26) can give rise to considerable numerical problems. The Neumann boundary condition

$$\frac{\partial \epsilon}{\partial y} = 0 \tag{27}$$

has been used in a variety of applications of the K-ϵ model (cf. Ref. 9). Although Eq. (27) is more computationally robust, it is completely ad hoc with no solid theoretical or experimental justification. In fact, recent direct numerical simulations of the Navier-Stokes equations for turbulent channel flow indicate that[5]

$$\frac{\partial}{\partial y^+}(\epsilon^+/\epsilon^+_{\text{wall}}) \doteq -0.25$$

at the wall; under such circumstances the use of Eq. (27) could lead to substantial errors.

The other major problem with the dissipation rate transport equation lies in the balance of terms at the wall. At a solid boundary, Eq. (6) reduces to

$$\nu\frac{\partial^2 \epsilon}{\partial y^2} = \Phi_\epsilon + \mathfrak{D}_\epsilon + \frac{\partial \epsilon}{\partial t} \tag{28}$$

For a fully developed turbulent boundary layer, $\partial\epsilon/\partial t = 0$ and $\mathfrak{D}_\epsilon \ll \Phi_\epsilon$ as discussed earlier; hence, Eq. (28) simplifies to

$$\nu\frac{\partial^2 \epsilon}{\partial y^2} = \Phi_\epsilon \tag{29}$$

Both Eqs. (28) and (29) have a major deficiency: the balance of terms at the wall involves higher-order correlations. This puts significant pressure on the accuracy of the near-wall modeling of the destruction of dissipation term that can further exacerbate the numerical stiffness problem.

On the other hand, the turbulent time scale $\tau \equiv K/\epsilon$ has a variety of natural boundary conditions. It is a simple matter to show that close to a wall

$$\tau \doteq \frac{y^2}{2\nu} \tag{30}$$

and, hence, at the wall

$$\tau = \frac{d\tau}{dy} = 0, \qquad \frac{d^2\tau}{dy^2} = \frac{1}{\nu} \tag{31}$$

Equations (30) and (31) have the advantage of being valid for *any* near-wall turbulence where the fluctuating velocity is expandable in a Taylor series. Furthermore, the balance at the wall in the transport equation for τ involves only exact viscous terms. This can be seen directly from the exact τ-transport equation, which takes the form

$$\frac{D\tau}{Dt} = \frac{\tau}{K}\tau_{ij}\frac{\partial \bar{u}_i}{\partial x_j} - 1 - \frac{\tau}{K}\mathfrak{D} - \frac{\tau^2}{K}\mathcal{P}_\epsilon + \frac{\tau^2}{K}\Phi_\epsilon + \frac{\tau^2}{K}\mathfrak{D}_\epsilon + \frac{2\nu}{K}\frac{\partial K}{\partial x_i}\frac{\partial \tau}{\partial x_i} - \frac{2\nu}{\tau}\frac{\partial \tau}{\partial x_i}\frac{\partial \tau}{\partial x_i} + \nu\nabla^2\tau \tag{32}$$

and hence, at the wall $y = 0$, the leading terms are

$$\frac{2\nu}{K}\frac{\partial K}{\partial y}\frac{\partial \tau}{\partial y} - \frac{2\nu}{\tau}\frac{\partial \tau}{\partial y}\frac{\partial \tau}{\partial y} + \nu\frac{\partial^2 \tau}{\partial y^2} - 1 = 0 \tag{33}$$

Here, each term on the left-hand side of Eq. (33) is $\mathcal{O}(1)$. The balance of terms in Eq. (33) is guaranteed if $\tau \doteq y^2/2\nu$ near the wall. Therefore, it appears that the two major problems with the K-ϵ model—namely, the lack of natural boundary conditions for ϵ and the appearance of higher-order correlations in the balance of terms at the wall—can be overcome by the use of a K-τ model.

Although the development of a two-equation turbulence model based on the turbulent time scale has been discussed in the literature (cf. Refs. 10–12), no systematic study of such models has been conducted for near-wall turbulence. Only the K-ω model, which is based on a modeled transport equation for the reciprocal turbulent time scale $\omega \equiv 1/\tau$, has been studied in these flows to any extent (see Refs. 3 and 4). In the K-ω model, a modeled transport equation for the reciprocal time scale ω is solved, which is in the form[4]

$$\frac{D\omega}{Dt} = C_{\omega 1}\frac{\omega}{K}\tau_{ij}\frac{\partial \bar{u}_i}{\partial x_j} - C_{\omega 2}\omega^2 + \frac{\partial}{\partial x_i}\left[\left(\nu + \frac{\nu_T}{\sigma_\omega}\right)\frac{\partial \omega}{\partial x_i}\right] \quad (34)$$

where $\nu_T = C_\mu K/\omega$, and $C_{\omega 1}$, $C_{\omega 2}$, σ_ω are constants that assume the values of 5/9, 5/6, and 2, respectively (again, $C_\mu = 0.009$). However, we will now show that Eq. (34) is inconsistent with the exact transport equation for ω near a wall: an exact viscous term should be added and $C_{\omega 2}$ should be damped. The exact transport equation for ω takes the form

$$\frac{D\omega}{Dt} = \frac{\mathcal{P}_\epsilon}{K} - \frac{\Phi_\epsilon}{K} - \frac{\mathfrak{D}_\epsilon}{K} - \frac{\omega\mathcal{P}}{K} + \omega^2 + \frac{\omega\mathfrak{D}}{K} + \frac{2\nu}{K}\frac{\partial\omega}{\partial x_i}\frac{\partial K}{\partial x} + \nu\nabla^2\omega \quad (35)$$

Hence, it is clear that the exact viscous cross-diffusion term, given by $(2\nu/K)(\partial\omega/\partial x_i)(\partial K/\partial x_i)$ in Eq. (35), has been neglected in the modeled ω-transport equation of Refs. 3 and 4 for simplicity. From Eq. (35), it is a simple matter to show that the leading terms in the near-wall balance of ω are as follows:

$$\frac{2\nu}{K}\frac{\partial\omega}{\partial y}\frac{\partial K}{\partial y} + \nu\frac{\partial^2\omega}{\partial y^2} + \omega^2 = 0 \quad (36)$$

at the plane solid boundary $y = 0$. Equation (36) is consistent with asymptotic solutions for K and ω that behave correctly near the wall, i.e.,

$$K \doteq \alpha y^2, \qquad \omega \doteq 2\nu/y^2 \quad (37)$$

In contrast to Eq. (36), the K-ω model of Wilcox based on Eq. (34) yields the balance of terms

$$\nu\frac{\partial^2\omega}{\partial y^2} - C_{\omega 2}\omega^2 = 0 \quad (38)$$

at the wall, which is incompatible with Eqs. (37) given that $C_{\omega 2} = 5/6$. Hence, the K-ω model of Wilcox yields asymptotically inconsistent solutions in near-wall turbulence (e.g., $K \sim y^{3.23}$; see Ref. 4).

The K-ω model can be made asymptotically consistent by the addition of the viscous cross-diffusion term $(2\nu/K)(\partial\omega/\partial x_i)(\partial K/\partial x_i)$ and by decomposition of $C_{\omega 2}$ as follows:

$$C_{\omega 2} = (11/6)f_2 - 1 \quad (39)$$

where f_2 is $\mathcal{O}(y^2)$ near the wall. However, we feel that it is preferable to derive a modeled transport equation for $\tau \equiv 1/\omega$, since τ is *not* singular near the wall.

New K-τ Model

Models for $\mathcal{P}_\epsilon$, Φ_ϵ, $\mathfrak{D}_\epsilon$, and $\mathfrak{D}$ are needed for closure of the τ-transport equation (32). The production-of-dissipation term will be modeled as it is in the K-ϵ model, i.e.,

$$\mathcal{P}_\epsilon = C_{\epsilon 1}\frac{\epsilon}{K}\tau_{ij}\frac{\partial \bar{u}_i}{\partial x_j} \equiv \frac{C_{\epsilon 1}}{\tau}\tau_{ij}\frac{\partial \bar{u}_i}{\partial x_j} \quad (40)$$

where $C_{\epsilon 1} = 1.44$. As mentioned earlier, this model is asymptotically consistent. The destruction-of-dissipation term Φ_ϵ will also be modeled similarly, i.e.,

$$\Phi_\epsilon = C_{\epsilon 2}f_2(\epsilon^2/K) \equiv C_{\epsilon 2}f_2(K/\tau^2) \quad (41)$$

Here, asymptotic consistence with Eq. (17) requires that f_2 be damped of $\mathcal{O}(y^2)$ near a wall. We will use a variation of the form for $C_{\epsilon 2}$ suggested by Hanjalic and Launder[13]:

$$C_{\epsilon 2} = 1.83[1 - (2/9)\exp(-Re_t^2/36)] \quad (42)$$

where $Re_t \equiv K^2/\nu\epsilon$ is the turbulence Reynolds number. Here, we set the high-turbulence Reynolds number value of $C_{\epsilon 2} = 1.83$ since it yields a somewhat more accurate value for the decay rate of grid turbulence than the traditional value of 1.92 (cf. Ref. 10). An exponential form is chosen for the wall damping function f_2 as follows:

$$f_2 = [1 - \exp(-y^+/A_2)]^2 \quad (43)$$

which is $\mathcal{O}(y^2)$ near the wall. Since at the wall

$$\frac{\partial^2\epsilon^+}{\partial y^{+2}} = C_{\epsilon 2}f_2\frac{\epsilon^{+2}}{K^+} \quad (44)$$

A_2 can be evaluated if $\partial^2\epsilon^+/\partial y^{+2}$, ϵ^+, and K^+ are known. By using the wall values of these quantities from direct numerical simulations of turbulent channel flow,[5] we obtain

$$A_2 \doteq 4.9 \quad (45)$$

The resulting model for Φ_ϵ is quite similar to that proposed recently by Myong and Kasagi.[14]

The turbulent diffusion term for τ is defined by

$$\mathfrak{D}_\tau = (\tau^2/K)\mathfrak{D}_\epsilon - (\tau/K)\mathfrak{D} \quad (46)$$

This term will be modeled by the gradient transport hypothesis

$$\mathfrak{D}_\tau = \frac{2\nu_T}{K\sigma_{\tau 1}}\frac{\partial K}{\partial x_i}\frac{\partial \tau}{\partial x_i} - \frac{2\nu_T}{\tau\sigma_{\tau 2}}\frac{\partial \tau}{\partial x_i}\frac{\partial \tau}{\partial x_i} + \frac{\partial}{\partial x_i}\left(\frac{\nu_T}{\sigma_{\tau 2}}\frac{\partial \tau}{\partial x_i}\right) \quad (47)$$

where ν_T is eddy viscosity and $\sigma_{\tau 1}$ and $\sigma_{\tau 2}$ are turbulent Prandtl numbers. In deriving Eq. (47), it has been assumed that the turbulent transport processes parallel the molecular ones (i.e., each turbulent transport term is coupled with a molecular diffusion term of the same general form). The turbulent Prandtl numbers for the last two terms on the right-hand side of Eq. (47) are taken to be equal so that this pairing of molecular and turbulent diffusion terms is also true if the reciprocal time scale ω is chosen as a variable instead of τ (the choice of ω as a variable instead of τ should not alter the basic physics of the model). Of course, the eddy viscosity is taken to be of the form

$$\nu_T = C_\mu f_\mu K\tau \quad (48)$$

where $C_\mu = 0.09$, and f_μ is a wall damping function that is $\mathcal{O}(1/y)$ near the wall. By an analysis of the two distinct effects of low-turbulence Reynolds number and near-wall proximity, Myong and Kasagi[14] proposed the model

$$f_\mu = (1 + 3.45/\sqrt{Re_t})[1 - \exp(-y^+/70)] \quad (49)$$

This model fits the experimental data[2] reasonably well with one exception—it asymptotes to one somewhat too slowly. Hence, we will consider the alternative model

$$f_\mu = (1 + 3.45/\sqrt{Re_t})\tanh(y^+/70) \quad (50)$$

since the hyperbolic tangent asymptotes to one faster by the necessary amount.

Now, for the purposes of clarity, we will summarize the K-τ model derived in this section:

$$\tau_{ij} = -\frac{2}{3}K\delta_{ij} + \nu_T\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) \quad (51)$$

$$\nu_T = C_\mu f_\mu K\tau \quad (52)$$

$$\frac{DK}{Dt} = \tau_{ij}\frac{\partial \bar{u}_i}{\partial x_j} - \frac{K}{\tau} + \frac{\partial}{\partial x_i}\left[\left(\nu + \frac{\nu_T}{\sigma_K}\right)\frac{\partial K}{\partial x_i}\right] \quad (53)$$

$$\frac{D\tau}{Dt} = (1 - C_{\epsilon 1})\frac{\tau}{K}\tau_{ij}\frac{\partial \bar{u}_i}{\partial x_j} + (C_{\epsilon 2}f_2 - 1) + \frac{2}{K}\left(\nu + \frac{\nu_T}{\sigma_{\tau 1}}\right)\frac{\partial K}{\partial x_i}\frac{\partial \tau}{\partial x_i}$$
$$- \frac{2}{\tau}\left(\nu + \frac{\nu_T}{\sigma_{\tau 2}}\right)\frac{\partial \tau}{\partial x_i}\frac{\partial \tau}{\partial x_i} + \frac{\partial}{\partial x_i}\left[\left(\nu + \frac{\nu_T}{\sigma_{\tau 2}}\right)\frac{\partial \tau}{\partial x_i}\right] \tag{54}$$

In Eqs. (51–54), $C_\mu = 0.09$ and $C_{\epsilon 1} = 1.44$, whereas $C_{\epsilon 2}, f_2$, and f_μ are given by Eqs. (42), (43), and (50), respectively. However, to complete the model, values for the turbulent Prandtl numbers $\sigma_{\tau 1}$, $\sigma_{\tau 2}$, and σ_K must be provided. In this regard, we first note that, if

$$\sigma_{\tau 1} = \sigma_{\tau 2} = \sigma_K = \sigma_\epsilon$$

then the modeled τ-transport equation (54) is equivalent to the ϵ-transport equation

$$\frac{D\epsilon}{Dt} = C_{\epsilon 1}\frac{\epsilon}{K}\tau_{ij}\frac{\partial \bar{u}_i}{\partial x_j} - C_{\epsilon 2}f_2\frac{\epsilon^2}{K} + \frac{\partial}{\partial x_i}\left[\left(\nu + \frac{\nu_T}{\sigma_\epsilon}\right)\frac{\partial \epsilon}{\partial x_i}\right] \tag{55}$$

Since the standard ϵ-transport equation (55) is known to perform well in several high-Reynolds-number turbulent flows, it is reasonable to believe that $\sigma_{\tau 1}$, $\sigma_{\tau 2}$, σ_K, and σ_ϵ must assume values that are reasonably close to one another. Furthermore, for local equilibrium flows with zero pressure gradient and logarithmic velocity profile, we must have [2]

$$C_{\epsilon 1} = C_{\epsilon 2} - (\kappa^2/\sigma_\epsilon\sqrt{C_\mu}) \tag{56}$$

where $\kappa \doteq 0.4$ is the von Kármán constant. Hence, for the values of $C_{\epsilon 1}$, $C_{\epsilon 2}$, and C_μ chosen herein, it follows that $\sigma_\epsilon \doteq 1.36$ and, hence,

$$\sigma_{\tau 1} \doteq \sigma_{\tau 2} \doteq \sigma_K \doteq 1.36 \tag{57}$$

It should be noted that the new modeled transport equation for τ given by Eq. (54) is equivalent to the ω-transport equation

$$\frac{D\omega}{Dt} = (C_{\epsilon 1} - 1)\frac{\omega}{K}\tau_{ij}\frac{\partial \bar{u}_i}{\partial x_j} - (C_{\epsilon 2}f_2 - 1)\omega^2$$
$$+ 2\left(\nu + \frac{\nu_T}{\sigma_{\tau 1}}\right)\frac{1}{K}\frac{\partial K}{\partial x_i}\frac{\partial \omega}{\partial x_i} + \frac{\partial}{\partial x_i}\left[\left(\nu + \frac{\nu_T}{\sigma_{\tau 2}}\right)\frac{\partial \omega}{\partial x_i}\right] \tag{58}$$

which differs from that of Refs. 3 and 4 by the presence of a cross-diffusion term and by the damping of the coefficient of ω^2 to one at the wall.

Calculations will be presented in the next section using the common value of 1.36 for $\sigma_{\tau 1}$, $\sigma_{\tau 2}$, and σ_K, which seems to be adequate for the present study. However, future research is needed to optimize these constants over a range of benchmark turbulent flows.

Comparison of Models

First, the performance of this new K-τ model will be examined for the incompressible flat-plate turbulent boundary layer at zero pressure gradient. Comparisons will be made initially with the K-ω model of Wilcox[4] and the K-ϵ model of Launder and Sharma[6] (a comparison with the K-ϵ model of Ref. 7 will be made later). The calculations to be presented were done with a two-dimensional boundary-layer code based on the implicit marching scheme of Edwards et al.[15] In the fully developed turbulent regime, approximately 100 grid points were used in the direction normal to the wall, with the first grid point at $y^+ \doteq 0.2$. The profiles of the turbulent fields to be discussed in the figures are for a Reynolds number $Re_\theta \doteq 1.6 \times 10^4$ based on the momentum thickness (this will allow for comparisons with the experimental data described by Patel et al.[2] which was compiled from a variety of sources, including Coles[16] and Schubauer[17]).

In Fig. 1, the predictions of the K-τ, K-ω, and K-ϵ models for the mean velocity are compared with experimental data.[2] It is clear that each model yields a logarithmic velocity profile for $30 < y^+ < 300$ that is well within the range of the experimental data. Furthermore, each model correctly yields $u^+ = y^+$ close to the wall (i.e., for $y^+ < 5$) and predicts the deviations from the law of the wall for $y^+ > 1000$. In Fig. 2, the Reynolds shear stress predicted by these three models is shown in lorarithmic coordinates. The predictions of the various models are exteremely close for $y^+ > 10$. However, for $y^+ < 10$, the differences between the model predictions are significant. Among these models, only the K-τ model yields a profile where $\overline{u'v'} \sim y^3$ for $y^+ < 10$, as indicated by experiments.[2] In Fig. 3, the predictions of the K-τ, K-ω, and K-ϵ models for the turbulent kinetic energy are compared. The K-τ model yields a peak in K^+ of approximately 4, which is well within the range of the experimental data[2] and the results of direct numerical simulations for turbulent channel flow.[5] Conversely, the K-ω model—as well as the K-ϵ model of Launder and Sharma—appears to yield peaks in the turbulent kinetic energy that are rather low. The turbulent kinetic energy near the wall is shown on a logarithmic plot in Fig. 4. Only the K-τ model yields $K \sim y^2$ for the entire interval $0 < y^+ < 10$; it

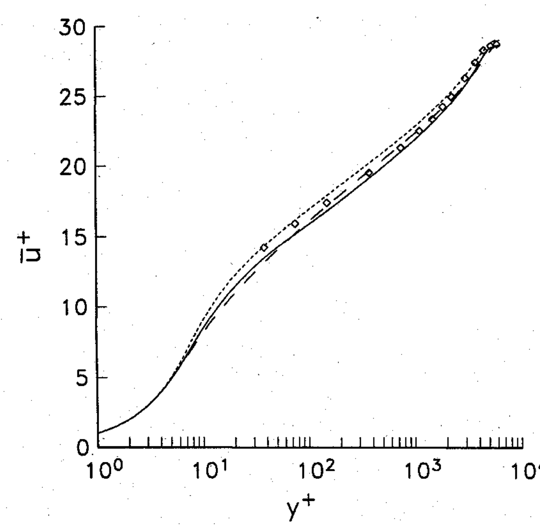

Fig. 1 Mean velocity profile predictions for flat-plate turbulent boundary layer ($Re_\theta \approx 1.6 \times 10^4$): ——, K-τ model; - - -, K-ω model; · · ·, K-ϵ model of Launder and Sharma[6]; ◇, experimental data.[2]

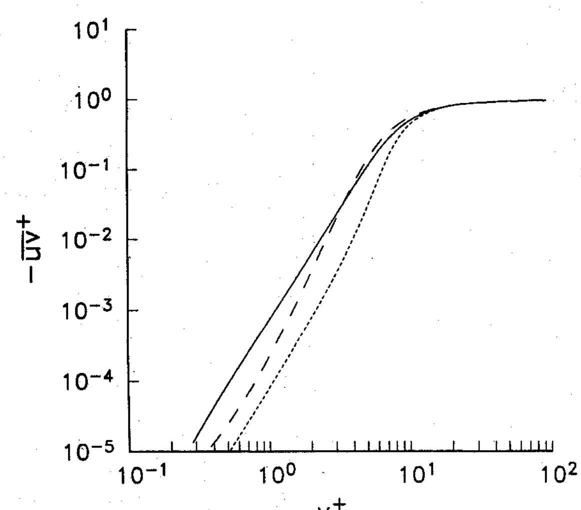

Fig. 2 Reynolds shear-stress profiles in logarithmic coordinates (legend same as Fig. 1).

yields the proportionality constant $\alpha^+ \doteq 0.05$—a result that is well within the range of the experimental data.

In Fig. 5, the profiles of the turbulent dissipation rate predicted by the K-τ, K-ω, and K-ϵ models are compared. Although the results are fairly close for $y^+ > 20$, there are some significant differences close to the wall. The K-τ model yields a value for the turbulent dissipation rate at the wall of $\epsilon_w^+ \doteq 0.1$, which is quite close to the value obtained from experiments.[2] Likewise, the peak in ϵ^+ is quite close to the value obtained from experiments.[2] In contrast to these results, the K-ω model and K-ϵ model of Launder and Sharma yield values for the wall dissipation ϵ_w^+ that are substantially too small. In Fig. 6, the variation of f_μ with y^+ is shown for these three models as well as for Chien's model.[7] Only the K-τ model is within the range of the experimental data.[2] We did not previously make comparisons with the results of Chien's K-ϵ model, since his model was calibrated by the experimental data for the flat-plate turbulent boundary layer at zero pressure gradient. However, the fact that this model has some inconsistencies can be seen in these results for f_μ.

The skin friction predicted by the K-τ model is shown as a function of the coordinate x along the plate in Fig. 7. It is clear that the results are in excellent agreement with the experimental data.[18] In Table 1, the fully developed skin friction and wall dissipation rate are tabulated for the four models considered in this study. Only the K-τ model and K-ϵ model of Chien

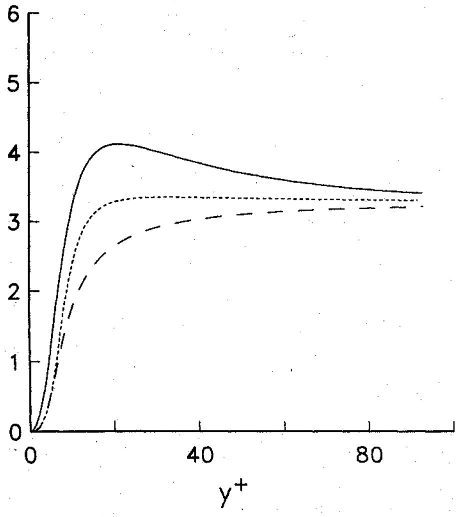

Fig. 3 Turbulent kinetic energy profiles (legend same as Fig. 1).

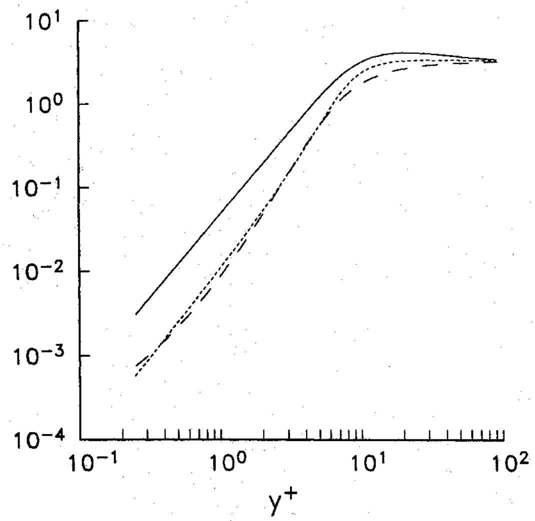

Fig. 4 Turbulent kinetic energy profiles in logarithmic coordinates (legend same as Fig. 1).

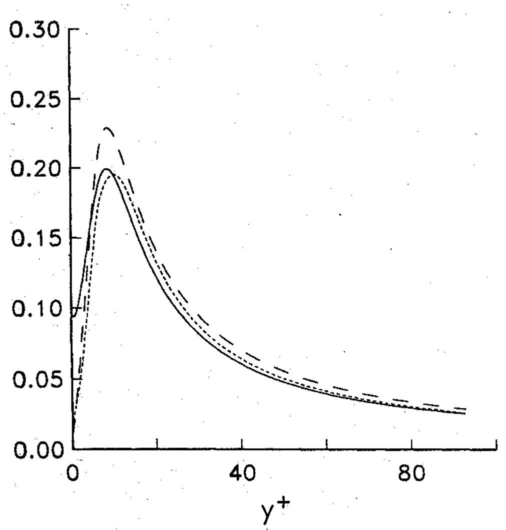

Fig. 5 Turbulent dissipation rate profiles (legend same as Fig. 1).

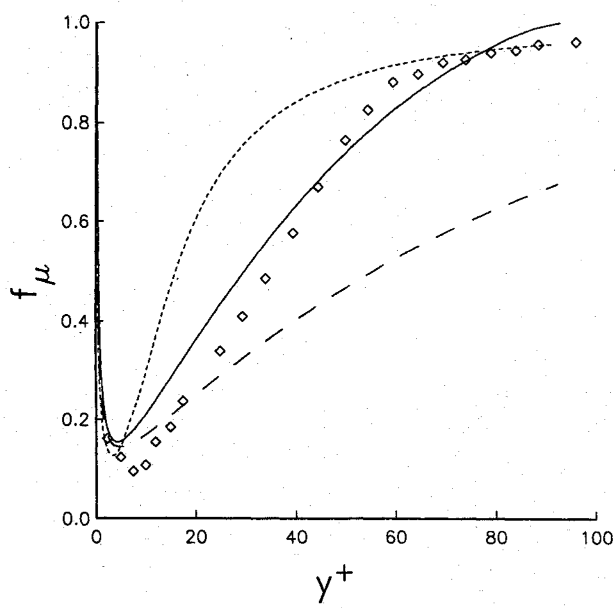

Fig. 6 Profiles of damping function f_μ for flat-plate turbulent boundary layer ($Re_\theta \approx 1.6 \times 10^4$): ——, K-τ model; · · ·, K-ϵ model of Launder and Sharma[6]; - - -, K-ϵ model of Chien[7]; ◇, experimental data.[2]

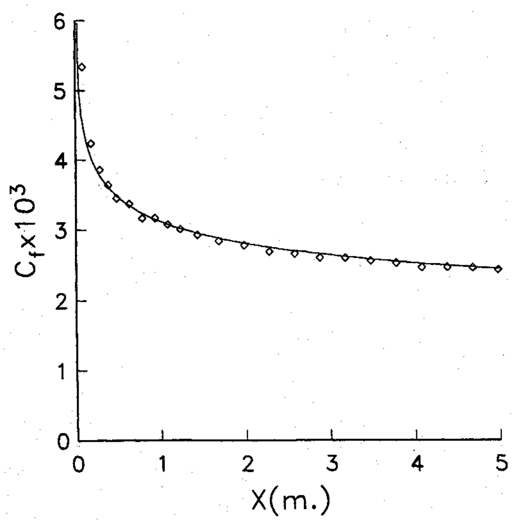

Fig. 7 Comparison of predictions of the K-τ model for skin friction with experimental data.[18]

Table 1 Comparison of model predictions for the wall dissipation rate ϵ_w^+ and skin friction C_f ($x = 4.987$ m, $Re_x = 1.156 \times 10^7$).

Model	ϵ_w^+	C_f
K-τ	0.094	0.00245
K-ϵ (Launder-Sharma)	0.021	0.00238
K-ϵ (Chien)	0.113	0.00244
K-ω	0	0.00244
Experiments 2, 18	0.05–0.10	0.00243

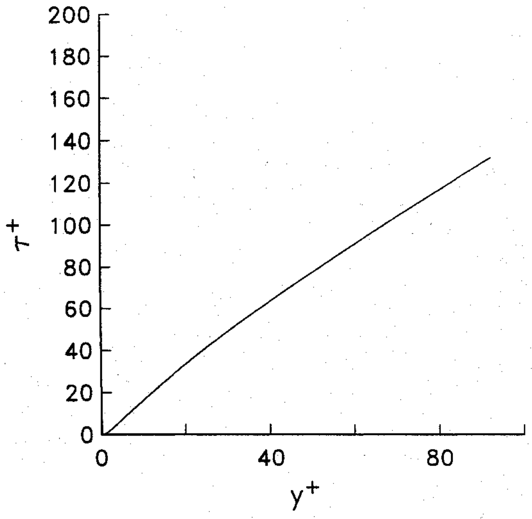

Fig. 8 Prediction of K-τ model for turbulent time-scale profile ($Re_\theta \approx 1.6 \times 10^4$).

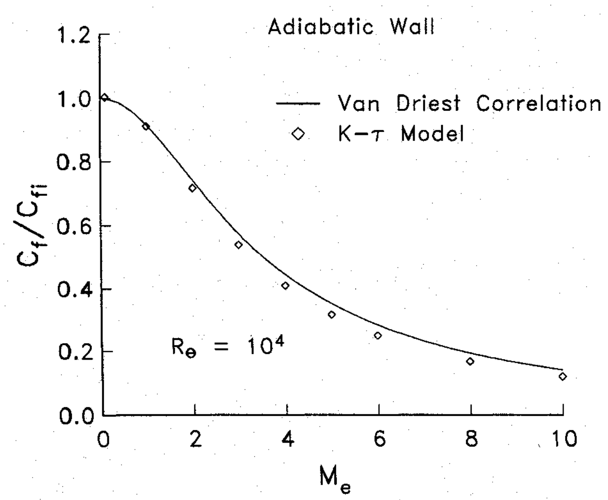

Fig. 9 Skin friction as a function of Mach number M_e for compressible flat-plate turbulent boundary layer: adiabatic wall ($Re_\theta = 10^4$, $C_{fi} \equiv [C_f]_{M_e=0}$).

yield results that are within the range of the experimental data. However, the more desirable features of τ as a variable instead of ϵ can be seen by a comparison of Fig. 8 with Fig. 5. It is clear from Fig. 8 that the turbulent time scale varies much more smoothly with the distance from the wall; its first derivative with respect to y *does not change sign.*

Finally, to demonstrate that the model works for more complex flow situations, we present results for the flat-plate boundary layer in the high-speed compressible regime. In Fig. 9, the normalized skin friction obtained from the K-τ model is compared with the Van Driest correlation,[19] for the case of the adiabatic wall; a range of Mach numbers $0 \leq M_e \leq 10$ are considered for $Re_\theta = 10^4$. In Fig. 10, the normalized skin friction obtained from the K-τ model is compared with the Van Driest correlation[19] for the case of surface cooling ($Re_\theta = 10^4$, $M_e = 5$). The results obtained for both cases are quite encouraging.

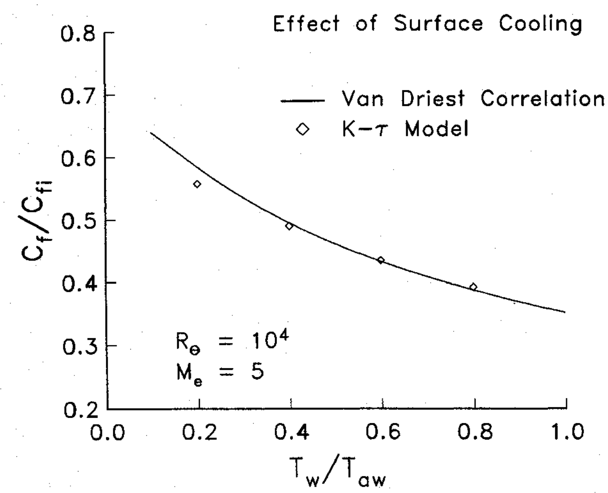

Fig. 10 Skin friction as a function of wall temperature T_w for compressible flat-plate turbulent boundary layer: effect of surface cooling ($Re_\theta = 10^4$, $M_e = 5$, $T_{aw} \equiv$ adiabatic wall temperature).

Conclusions

A basic theoretical and computational study of two-equation models for near-wall turbulent flows has been conducted. The major findings of this study can be summarized as follows:

1) The K-ω model of Wilcox and co-workers[3,4] neglects an exact viscous cross-diffusion term and does not damp the destruction-of-dissipation term near a wall. These two simplifications give rise to asymptotically incorrect solutions for the turbulent kinetic energy ($K \sim y^{3.23}$) near a solid boundary. However, for many applications, the K-ω model does yield adequate predictions for the skin friction and the turbulence statistics away from the wall.

2) The K-ϵ model can be made asymptotically consistent by the satisfaction of two constraints: the coefficient of the destruction-of-dissipation term must be damped of $\mathcal{O}(y^2)$ near a wall, and the coefficient in the eddy viscosity must be damped of $\mathcal{O}(1/y)$ near a wall. Most existing corrections to the K-ϵ model yield poor results in near-wall turbulent flows due to the violation of these constraints.

3) There are numerical stiffness problems with the K-ϵ model due to the lack of natural boundary conditions for the dissipation rate and the fact that the balance of terms for the dissipation at the wall involves unknown higher-order correlations that need to be modeled. These problems can, to a large extent, be overcome by the use of the turbulent time scale $\tau \equiv K/\epsilon$ as a variable since τ has natural boundary conditions arising from the no-slip condition and since the wall balance for τ only involves exact viscous terms.

4) A new K-τ model was developed by making use of these ideas combined with improved variations of the wall damping functions f_2 and f_μ recently developed by Myong and Kasagi.[14] A preliminary test of this K-τ model for the turbulent flat-plate boundary layer yielded results that are quite encouraging. However, further tests and possible refinements are required before more definitive conclusions can be drawn.

In future research, the K-τ model will be subjected to more stringent tests involving adverse pressure gradients and high-speed compressible flows with separation.

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